Cohomological characterization of Universal bundles of the Grassmannian of lines

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Motivation

- Horrock's criterion $\rightsquigarrow$ *A vector bundle F over $\mathbb{P}^n$ splits if and only if F does not have intermediate cohomology*
- Ottaviani $\rightsquigarrow$ *splitting criterion for $\mathbb{G}(k,n)$ and quadrics*
- Arrondo and Graña $\rightsquigarrow$ *characterized* $\oplus\mathcal{O}(l_{i_0})\bigoplus\oplus\mathcal{Q}(l_{i_1})$ $(\mathbb{G}(1,4))$
- Costa and Miró-Roig $\rightsquigarrow$ *characterized* $\mathbb{S}_\lambda\mathcal{Q}$ $(\mathbb{G}(k,n))$
- Arrondo and Malaspina $\rightsquigarrow$ *gave an improvement of the splitting criterion for the Grassmannian of lines*

1. Notation $\rightsquigarrow \mathcal{Q}$, $\mathcal{S}$, universal exact sequence, E-N complex
2. Splitting criterion for Grassmannian of lines
   - Ideas (Serre duality, E-N complex)
   - Result
   - Comparison
3. Next Step $\rightsquigarrow \mathcal{O}$, $\mathcal{Q}$
   - Ideas (Induction on cohomology)
   - Sketch of the proof
   - Result
   - Comparison for $\mathbb{G}(1,4)$
4. General Step $\rightsquigarrow \mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, $\ldots$, $S^k\mathcal{Q}$ with $k \leq n-2$ (Chapter 3)
   - Ideas (Double induction)
   - Sketch of the proof
   - Result
5. Derived categories vs E-N complex + Serre Duality (Chapter 4)

Notation for $\mathbb{G}(1, n)$:

- $\mathbb{G}(1, n) = \{1-\text{dimensional subspaces of } \mathbb{P}^n = \mathbb{P}(V)\}$
- $\mathcal{Q}^\vee = \{(v, \Lambda) \in V^* \times \mathbb{G}(1, n) \mid v \in \Lambda\}$
- The universal exact sequence is:

$$0 \longrightarrow \mathcal{Q}^\vee \longrightarrow V^* \otimes \mathcal{O} \xrightarrow{\rho} \mathcal{S} \longrightarrow 0$$

- $\mathcal{Q}^\vee$ is the universal bundle of rank 2
- $\mathcal{S}$ is the universal bundle of rank $n - 1$
- By making $\wedge^j \rho$ we get some Eagon-Northcott complexes

The following are equivalent:

- $\mathcal{O}$ is a direct summand of F
- there exist maps $\mathcal{O} \longrightarrow F$ and $F \longrightarrow \mathcal{O}$ whose composition is non zero

We can relate the composition:

$$Hom(\mathcal{O}, F) \times Hom(F, \mathcal{O}) \longrightarrow Hom(\mathcal{O}, \mathcal{O})$$

$$(H^0(F) \times H^0(F^\vee) \longrightarrow H^0(\mathcal{O}))$$

with the perfect pairing giving by Serre's duality.

We can get the following commutative diagram:

$$\begin{array}{rcl}
H^{n-1}(F \otimes S^{n-1}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-1}\mathcal{Q}^\vee(-1)) & \stackrel{\phi}{\longrightarrow} & H^{2n-2}(\mathcal{O}(-n-1)) \\
\uparrow id \times \psi_2 & \circlearrowleft & \psi_4 \uparrow \simeq \\
H^{n-1}(F \otimes S^{n-1}\mathcal{Q}(-n)) \times H^0(F^\vee) & \longrightarrow & H^{n-1}(S^{n-1}\mathcal{Q}(-n)) \\
\uparrow \psi_1 \times id & \circlearrowleft & \psi_3 \uparrow \simeq \\
H^0(F) \times H^0(F^\vee) & \stackrel{\phi'}{\longrightarrow} & H^0(\mathcal{O})
\end{array}$$

We use the following Eagon-Northcott complex to build the injective map $\psi_1 : H^0(F) \longrightarrow H^{n-1}(F \otimes S^{n-1}\mathcal{Q}(-n))$:

$$0 \rightarrow S^{n-1}\mathcal{Q}(-n) \rightarrow V^* \otimes S^{n-2}\mathcal{Q}(-n+1) \rightarrow \bigwedge^2 V^* \otimes S^{n-3}\mathcal{Q}(-n+2) \rightarrow \dots$$
$$\dots \rightarrow \bigwedge^{n-2} V^* \otimes \mathcal{Q}(-2) \rightarrow \bigwedge^{n-1} V^* \otimes \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0$$

And the following complex to build the surjective map $\psi_2 : H^0(F^\vee) \longrightarrow H^{n-1}(F^\vee \otimes S^{n-1}\mathcal{Q}^\vee(-1))$:

$$0 \rightarrow S^{n-1}\mathcal{Q}^\vee(-1) \rightarrow V^* \otimes S^{n-2}\mathcal{Q}^\vee(-1) \rightarrow \bigwedge^2 V^* \otimes S^{n-3}\mathcal{Q}^\vee(-1) \rightarrow \dots$$
$$\dots \rightarrow \bigwedge^{n-2} V^* \otimes \mathcal{Q}^\vee(-1) \rightarrow \bigwedge^{n-1} V^* \otimes \mathcal{O}(-1) \rightarrow \mathcal{O} \rightarrow 0$$

Step 0

(Splitting Criterion) A vector bundle F over $\mathbb{G}(1,n)$ splits if and only if $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ where $(i,j) \in A_0 \cup B_0$.

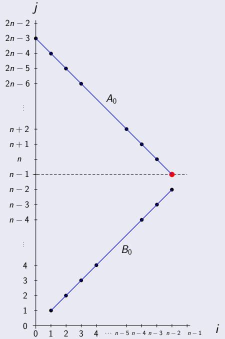

Figure: AM's splitting critetion

First we compare with the splitting criterion made by Ottaviani.

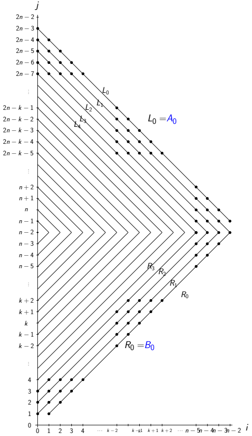

Figure: Ottaviani's splitting criterion

Now we compare with the splitting criterion obtained by using derived categories.

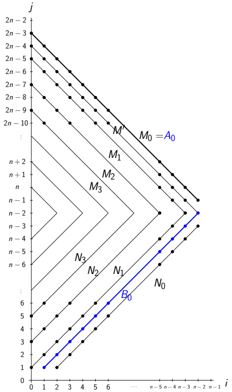

Figure: Splitting critetion with derived categories

Goal: Characterize direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$.

As before the following are equivalent:

- $\mathcal{Q}$ is a direct summand of F
- there exist maps $\mathcal{Q} \longrightarrow F$ and $F \longrightarrow \mathcal{Q}$ whose composition is non zero

We can relate the composition:

$$Hom(\mathcal{Q}, F) \times Hom(F, \mathcal{Q}) \longrightarrow Hom(\mathcal{Q}, \mathcal{Q})$$

$$(H^0(F \otimes \mathcal{Q}^\vee) \times H^0(F^\vee \otimes \mathcal{Q}) \longrightarrow H^0(\mathcal{Q}^\vee \otimes \mathcal{Q}))$$

with the perfect pairing giving by Serre's duality.

We can get the following commutative diagram:

$$\begin{array}{ccc} H^{n-1}(F \otimes S^{n-2}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-2}\mathcal{Q}^\vee(-1)) & \xrightarrow{\phi} & H^{2n-2}(\mathcal{O}(-n-1)) \\ \uparrow \psi_1 \times id & \circlearrowleft & \psi_3 \uparrow \simeq \\ H^0(F \otimes \mathcal{Q}^\vee) \times H^{n-1}(F^\vee \otimes S^{n-2}\mathcal{Q}^\vee(-1)) & \longrightarrow & H^{n-1}(\mathcal{Q}^\vee \otimes S^{n-2}\mathcal{Q}^\vee(-1)) \\ \uparrow id \times \psi_2 & \circlearrowleft & \psi_4 \uparrow \simeq \\ H^0(F \otimes \mathcal{Q}^\vee) \times H^0(F^\vee \otimes \mathcal{Q}) & \xrightarrow{\phi'} & H^0(\mathcal{Q}^\vee \otimes \mathcal{Q}) \end{array}$$

We can build the natural surjective maps ψ_1 and ψ_2 by using some particular Eagon-Northcott complexes.

For $\psi_1 : H^0(F \otimes \mathcal{Q}^\vee) \longrightarrow H^{n-1}(F \otimes S^{n-2}\mathcal{Q}(-n))$ we use:

$$0 \longrightarrow S^{n-2}\mathcal{Q}(-n) \longrightarrow V^* \otimes S^{n-3}\mathcal{Q}(-n+1) \longrightarrow \bigwedge^2 V^* \otimes S^{n-4}\mathcal{Q}(-n+2) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-3} V^* \otimes \mathcal{Q}(-3) \longrightarrow \bigwedge^{n-2} V^* \otimes \mathcal{O}(-2) \longrightarrow V \otimes \mathcal{O}(-1) \longrightarrow \mathcal{Q}^\vee \longrightarrow 0$$

And for $\psi_2 : H^0(F^\vee \otimes \mathcal{Q}) \longrightarrow H^{n-1}(F^\vee \otimes S^{n-2}\mathcal{Q}^\vee(-1))$ we use:

$$0 \longrightarrow S^{n-2}\mathcal{Q}^\vee(-1) \longrightarrow V^* \otimes S^{n-3}\mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^2 V^* \otimes S^{n-4}\mathcal{Q}^\vee(-1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-3} V^* \otimes \mathcal{Q}^\vee(-1) \longrightarrow \bigwedge^{n-2} V^* \otimes \mathcal{O}(-1) \longrightarrow V \otimes \mathcal{O} \longrightarrow \mathcal{Q} \longrightarrow 0$$

If we only suppose the vanishing of cohomology that make ψ_1 and ψ_2 surjective maps we are just characterizing direct sums of twists of $\mathcal{Q}$. The hypotheses are $H^j_*(F \otimes S^i \mathcal{Q}) = 0$ with (i, j) in the following figure.

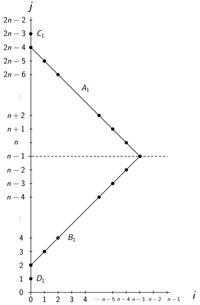

Figure: Characterize direct sums of twists of $\mathcal{Q}$

Ideas to characterize direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$:

- Remove one hypothesis of **Step 0** ($H^{n-1}_*(F \otimes S^{n-2}\mathcal{Q}) \neq 0$)
- **Our conditions**: the remaining hypotheses of Step 0 and the hypotheses that characterize direct sums of twists of $\mathcal{Q}$
- Induction on $\sum_l h^{n-1}(F \otimes S^{n-2}\mathcal{Q}(l)) = m$.
    - $m = 0 \Rightarrow$ Step 0
    - we suppose the result true for $m - 1$
    - we prove the result for $m \neq 0$

Sketch of the proof:

- There exists an l such that $H^{n-1}(F \otimes S^{n-2}\mathcal{Q}(l)) \neq 0$ (we choose $l = -n$)
- We obtain the commutative diagram $\Rightarrow$ F has as a direct summand $\mathcal{Q}$:
$$F = \mathcal{Q} \oplus F'$$
- F' satisfies the same hypotheses of F and
$$m' := \sum_l h^{n-1}(F' \otimes S^{n-2}\mathcal{Q}(l)) = m - 1$$
- Applying induction hypothesis to F' $\Rightarrow$ F can be expressed as direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$

Step 1

A vector bundle F over $\mathbb{G}(1,n)$ *can be expressed as direct sums of twists of* $\mathcal{O}$ *and* $\mathcal{Q}$ *if and only if* $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ *where* (i,j) *are the points in the following figure.*

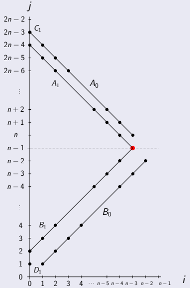

Figure: Characterize direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$

We can compare this result with the one given by E. Arrondo and B. Graña for the case $\mathbb{G}(1,4)$. Observe that our characterization has one less condition.

A vector bundle F over $\mathbb{G}(1,4)$ can be expressed as direct sums of twists of $\mathcal{O}$ and $\mathcal{Q}$ if and only if $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ where the points (i,j) are in the following figure.

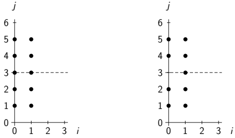

Figure: AG characterization vs Step 1

Goal: Characterize direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, ..., $S^k\mathcal{Q}$ with $k \leq n-2$.

The following are equivalent:

- $S^k\mathcal{Q}$ is a direct summand of F
- there exist maps $S^k\mathcal{Q} \longrightarrow F$ and $F \longrightarrow S^k\mathcal{Q}$ whose composition is non zero

We can relate the composition:

$$\begin{array}{rcl} Hom(S^k\mathcal{Q}, F) \times Hom(F, S^k\mathcal{Q}) & \rightarrow & Hom(S^k\mathcal{Q}, S^k\mathcal{Q}) \\ (H^0(F \otimes S^k\mathcal{Q}^\vee) \times H^0(F^\vee \otimes S^k\mathcal{Q}) & \rightarrow & H^0(S^k\mathcal{Q}^\vee \otimes S^k\mathcal{Q})) \end{array}$$

with the perfect pairing giving by Serre's duality.

We can get the following commutative diagram:

$$\begin{array}{rcl}
H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n)) \times H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) & \xrightarrow{\phi} & H^{2n-2}(\mathcal{O}(-n-1)) \\
\uparrow \psi_1 \times id & \circlearrowleft & \psi_3 \uparrow \simeq \\
H^0(F \otimes S^k\mathcal{Q}^\vee) \times H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) & \longrightarrow & H^{n-1}(S^k\mathcal{Q}^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1)) \\
\uparrow id \times \psi_2 & \circlearrowleft & \psi_4 \uparrow \simeq \\
H^0(F \otimes S^k\mathcal{Q}^\vee) \times H^0(F^\vee \otimes S^k\mathcal{Q}) & \xrightarrow{\phi'} & H^0(S^k\mathcal{Q}^\vee \otimes S^k\mathcal{Q})
\end{array}$$

We can build the natural surjective maps ψ_1 and ψ_2 by using some particular Eagon-Northcott complexes.

For $\psi_1 : H^0(F \otimes S^k\mathcal{Q}^\vee) \longrightarrow H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(-n))$ we use:

$$0 \to S^{n-k-1}\mathcal{Q}(-n) \to V^* \otimes S^{n-k-2}\mathcal{Q}(-n+1) \to \bigwedge^2 V^* \otimes S^{n-k-3}\mathcal{Q}(-n+2) \to \ldots$$

$$\ldots \to \bigwedge^{n-k-2} V^* \otimes \mathcal{Q}(-k-2) \to \bigwedge^{n-k-1} V^* \otimes \mathcal{O}(-k-1) \to$$

$$\to \bigwedge^k V \otimes \mathcal{O}(-k) \to \bigwedge^{k-1} V \otimes \mathcal{Q}(-k) \to \ldots$$

$$\ldots \to \bigwedge^2 V \otimes S^{k-2}\mathcal{Q}(-k) \to V \otimes S^{k-1}\mathcal{Q}(-k) \to S^k\mathcal{Q}(-k) \to 0$$

And for $\psi_2 : H^0(F^\vee \otimes S^k\mathcal{Q}) \longrightarrow H^{n-1}(F^\vee \otimes S^{n-k-1}\mathcal{Q}^\vee(-1))$ we use:

$$0 \to S^{n-k-1}\mathcal{Q}^\vee(-1) \to V^* \otimes S^{n-k-2}\mathcal{Q}^\vee(-1) \to \bigwedge^2 V^* \otimes S^{n-k-3}\mathcal{Q}^\vee(-1) \to \dots$$

$$\dots \to \bigwedge^{n-k-2} V^* \otimes \mathcal{Q}^\vee(-1) \to \bigwedge^{n-k-1} V^* \otimes \mathcal{O}(-1) \to$$

$$\to \bigwedge^k V \otimes \mathcal{O} \to \bigwedge^{k-1} V \otimes \mathcal{Q} \to \dots$$

$$\dots \to \bigwedge^2 V \otimes S^{k-2}\mathcal{Q} \to V \otimes S^{k-1}\mathcal{Q} \to S^k\mathcal{Q} \to 0$$

If we only suppose the vanishing of cohomology that make ψ_1 and ψ_2 surjective maps we are just characterizing direct sums of twists of $S^k\mathcal{Q}$. The hypotheses are $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ with (i,j) in the following figure.

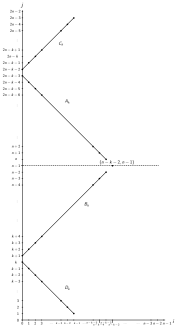

Figure: Characterize direct sums of twists of $S^k\mathcal{Q}$

Ideas to characterize direct sums of $\mathcal{O}, \mathcal{Q}, S^2\mathcal{Q}, \ldots, S^k\mathcal{Q}$:

- Remove one hypothesis of **Step k-1** $(H_*^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}) \neq 0)$
- **Our conditions**: the remaining hypotheses of Step $k-1$ and the hypotheses that characterize direct sums of twists of $S^k\mathcal{Q}$
- Induction on $\sum_l h^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(l)) = m$.
  - $m = 0 \Rightarrow$ Step $k-1$
  - we suppose the result true for $m-1$
  - we prove the result for $m \neq 0$

Sketch of the proof:

- There exists an l such that $H^{n-1}(F \otimes S^{n-k-1}\mathcal{Q}(l)) \neq 0$ (we choose $l = -n$)
- We obtain the commutative diagram $\Rightarrow$ F has as a direct summand $S^k\mathcal{Q}$:

$$F = S^k\mathcal{Q} \oplus F'$$

- F' satisfies the same hypotheses of F and

$$m' := \sum_l h^{n-1}(F' \otimes S^{n-k-1}\mathcal{Q}(l)) = m - 1$$

- Applying induction hypothesis to F' $\Rightarrow$ F can be expressed as direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, $\ldots$, $S^{k-1}\mathcal{Q}$ and $S^k\mathcal{Q}$

Step k

A vector bundle F over $\mathbb{G}(1,n)$ can be expressed as direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, . . ., $S^{k-1}\mathcal{Q}$ and $S^k\mathcal{Q}$ with $k \leq n-2$ if and only if $H^j_*(F \otimes S^i\mathcal{Q}) = 0$ where (i,j) are the points in the following figure.

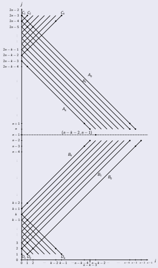

Figure: Characterize direct sums of twists of $\mathcal{O}$, $\mathcal{Q}$, $S^2\mathcal{Q}$, . . ., $S^k\mathcal{Q}$

To use derived categories we consider the following resolution of the diagonal $\Delta \subseteq X \times X$ where $X = \mathbb{G}(k, n)$:

$$0 \to \bigwedge^{(k+1)(n-k)}(\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) \to \dots \bigwedge^2(\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) \to \mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee \to \mathcal{O}_{X\times X} \to \mathcal{O}_\Delta \to 0$$

These elements decompose in the following way:

$$\bigwedge^r(\mathcal{Q}^\vee \boxtimes \mathcal{S}^\vee) = \bigoplus_{|\lambda|=r} \mathbb{S}_\lambda \mathcal{Q}^\vee \boxtimes \mathbb{S}_{\lambda'} \mathcal{S}^\vee$$

where the sum goes over all Young tableau with r cells, λ' is the conjugate Young tableau and $\mathbb{S}_\lambda$ is the Schur functor associated to the tableau λ

Observe that $H^{2n-2}(\mathcal{O}(-n-1)) = Ext^{2n-2}(\mathcal{O}, \mathcal{O}(-n-1))$ and the element that generates is precisely:

$$0 \longrightarrow \mathcal{O}(-n-1) \longrightarrow \bigwedge^{n-1} V \otimes \mathcal{O}(-n-2) \longrightarrow \bigwedge^{n-2} V \otimes \mathcal{Q}(-n) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{2} V \otimes S^{n-3}\mathcal{Q}(-n) \longrightarrow V \otimes S^{n-2}\mathcal{Q}(-n) \longrightarrow V^* \otimes S^{n-2}\mathcal{Q}(-n+1) \longrightarrow \dots$$

$$\dots \longrightarrow \bigwedge^{n-2} V^* \otimes \mathcal{Q}(-2) \longrightarrow \bigwedge^{n-1} V^* \otimes \mathcal{O}(-1) \longrightarrow \mathcal{O} \longrightarrow 0$$

Conclusion:

The resolution of the diagonal has more pieces than the previous complex.

Thank you!