VECTOR BUNDLES AND GRASSMANNIANS

1. Grassmannian

The goal of my research is to give a particular expresion of the normal bundle of the grassmannian $\mathbb{G}(1,n)$. So, I must start talking about some definitions and properties of grassmannians.

Suppose a vector space V (over a field that could be $\mathbb{C}$) of dimension $n+1$. We will denote:

- $\mathbb{P}^n = \mathbb{P}(V)$ the projective space of all the hiperplanes of V
- $\mathbb{G}(k,n) = k-$dimensional linear subspaces of $\mathbb{P}^n$ or equivalently $(k+1)-$dimensional linear subspaces of V^*.

$\mathbb{G}(k,n)$ is a variety because is covered by affine charts. To prove this we fix $\Lambda \in \mathbb{G}(k,n)$ an element, that is represented by a matrix (Plücker matrix)

$$\Lambda = \begin{pmatrix} a_{00} & \dots & a_{0n} \\ \vdots & & \vdots \\ a_{k0} & \dots & a_{kn} \end{pmatrix}$$

where the rows are the coordinates of a basis of Λ. Of course this representation is not unique. If we change the basis of Λ, the matrix changes by multiplying on the left by the non-degenerate square matrix of order $k+1$ corresponding to the change of basis in Λ. Assume for instance that the minor corresponding to the first $k+1$ columns is not zero. This menas that after multiplying by a suitable matrix one can represent Λ in a unique way by the matrix

$$\begin{pmatrix} 1 & 0 & \dots & 0 & b_{0k+1} & \dots & b_{0n} \\ 0 & 1 & \dots & 0 & b_{1k+1} & \dots & b_{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \dots & 1 & b_{kk+1} & \dots & b_{kn} \end{pmatrix}$$

Hence $\mathbb{G}(k,n)$ contains an open affine subset of dimension $(k+1)(n-k)$ (of coordenates $b_{0k+1}, \dots, b_{kn}$).Since at least one of the minors of order $k+1$ of the Plücker matrix is not zero, $\mathbb{G}(k,n)$ can be covered by $\binom{n+1}{k+1}$ affine pieces. It is very easy (but very tedious to write) to describe the change of coordenates from one piece to another. In conclusion we have that $\mathbb{G}(k,n)$ can be viewed as an abstract manifolds of dimension $(k+1)(n-k)$.

In order to view $\mathbb{G}(k,n)$ as a projective variety, one needs to consider the so-called *Plücker embedding*

$$\begin{array}{rccc} \varphi_{kn}: & \mathbb{G}(k,\mathbb{P}(V)) & \longrightarrow & \mathbb{P}(\bigwedge^{k+1} V^*) \\ & L[v_0,\dots,v_k] & \longmapsto & [v_0 \wedge \dots \wedge v_k] \end{array}$$

Here $L[v_0,\dots,v_k]$ represents the linear span in $\mathbb{P}(V)$ of the points represented by the independent vectors $v_0,\dots,v_k \in V^*$ and $[v_0 \wedge \dots \wedge v_k]$ menas the point of $\mathbb{P}(\bigwedge^{k+1} V)$ represented by $v_0 \wedge \dots \wedge v_k$. It is easy to check that φ_{kn} is well defined and it is not very hard to see that φ_{kn} provides an embedding of $\mathbb{G}(k,n)$ in $\mathbb{P}(\bigwedge^{k+1} V)$ as an algebraic subvariety.

2. Universal bundles

Now we will talk about universal bundles. Fix $\Lambda \in \mathbb{G}(k,n)$ as a subspace of V^*. We can consider

$$\begin{array}{c} Q^* = \{(\Lambda, v) \in \mathbb{G} \times V^* | v \in \Lambda\} \subseteq \mathbb{G} \times V^* \\ \downarrow q \\ \mathbb{G} \end{array}$$

Each fiber of $\mathbb{G}$ has dimension $k+1$, that way Q^* is a bundle of rank $k+1$.

The map q provides Q^* a natural structure of vector bundle over $\mathbb{G}$. And more precisely, it is a vector subbundle of the trivial bundle $\mathbb{G} \times V^*$. Hence we can also consider the quotient vector bundle, which we will call S.

$$S := \mathbb{G} \times V^*/Q^*$$

By dualazing we get the so-called *universal exact sequence* on $\mathbb{G}$:

$$0 \longrightarrow \mathcal{S}^* \longrightarrow V \otimes \vartheta_{\mathbb{G}} \longrightarrow \mathcal{Q} \longrightarrow 0$$

Definition 2.1. *The sheaves $\mathcal{S}^*$ and $\mathcal{Q}$ appearing in the sequence are called respectively the universal subbundle ande the universal quotient bundle. Note that they have respective ranks $n-k$ and $k+1$.*

With all this concepts we can give a kind of motivation for the goal of the research. We want to study $\mathbb{G}(1,n)$, by the Plücker embedding we have $\mathbb{G}(1,n) \subseteq \mathbb{P}(\bigwedge^2 V) = \mathbb{P}^{\binom{n+1}{2}-1}$ We also havet that the rank of N must be:

$$dim(N) = dim(\mathbb{P}) - dim(\mathbb{G}) = \frac{n^2+n-2}{2} - (2n-2) = \frac{n^2-3n+2}{2} =$$
$$\frac{(n-1)(n-2)}{2} = \binom{n-1}{2}$$

that is the rank of $\bigwedge^2 S$.

Two properties of universal bundles for $\mathbb{G}(1,n)$ are:

- locally $Q^* \otimes S^* \cong \Omega_{\mathbb{G}}$
- $Q \otimes \vartheta_{\mathbb{G}}(-1) \cong Q^*$

3. Normal Bundle

Now we will start talking about the normal bundle. Suppose $\mathbb{P}^n$, $\Omega_{\mathbb{P}^n}$ and $\mathbb{G}(1,n) \subseteq \mathbb{P}^n$. By a proposition we can say that there exist an epimorphisim $\Omega_{\mathbb{P}^n|\mathbb{G}} \xrightarrow{\varphi} \Omega_{\mathbb{G}}$ that consist in the restriction of the differentials.

Definition 3.1. *We call conormal bundle ($N^*_{\mathbb{G}|\mathbb{P}^n}$) of $\mathbb{G}$ in $\mathbb{P}^n$ to the ker of the previous epimorphisim φ. We call normal bundle of $\mathbb{G}$ in $\mathbb{P}^n$ to the dual $N_{\mathbb{G}|\mathbb{P}^n}$ of the conormal bundle.*

Hence, we have the following exact sequence

$$0 \to N^*_{\mathbb{G}|\mathbb{P}} \to \Omega_{\mathbb{P}|\mathbb{G}} \to \Omega_{\mathbb{G}} \to 0$$

This is one of the sequences we have to use to build our diagram.

And now we will enumerate some exact sequences that will be needed for the space $\mathbb{G}(1,n)$.

- Universal Exact Sequences:

$$0 \to S^* \to H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}} \to Q \to 0$$

we shall modificate this until we get the sequence we need,

$$0 \to S^*(-1) \to H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) \to Q(-1) = Q^* \to 0$$

$$0 \to S^* \otimes S^*(-1) \to S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) \to S^* \otimes Q^* \to 0$$

For notation we also know that $S^* \otimes S^*(-1) = S^* \otimes S^* \otimes \vartheta(-1)$.

- Eagon-Northcott Complex: suposse we have the following exact sequence of vectorial spaces

$$0 \longrightarrow V'^* \xrightarrow{\varphi^*} V^* \xrightarrow{\beta^*} V''^* \longrightarrow 0$$

and suposse also that $dimV'' = 2$ then we have that this sequence

$$0 \longrightarrow S^2V'^* \longrightarrow V'^* \otimes V^* \longrightarrow \bigwedge^2 V^* \longrightarrow \bigwedge^2 V''^* \longrightarrow 0$$

is exact. I have proved this in the lengauge of homomorphisim instead of in language of tensor product. But the maps that we finally get are: (for example $V'^* \otimes V^* \to \bigwedge^2 V^*$)

$$\alpha' : V' \to \mathbb{K}$$

$$\lambda : V \to \mathbb{K}$$

$$\begin{array}{rcl} V' \times V & \longrightarrow & \mathbb{K} \\ (v', v) & \longmapsto & \alpha'(v')\lambda(v) \end{array}$$

$$\begin{array}{rcl} V \times V & \longrightarrow & \mathbb{K} \\ (v_1, v_2) & \longmapsto & \alpha' \circ \varphi(v_1)\lambda(v_2) - \alpha' \circ \varphi(v_2)\lambda(v_1) \end{array}$$

so finally

$$\begin{array}{rcl} V'^* \otimes V^* & \longrightarrow & \bigwedge^2 V^* \\ (\alpha', \lambda) & \longmapsto & \varphi^*(\alpha') \wedge \lambda \end{array}$$

If we begin from the universal exact sequence

$$0 \to S^* \to H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}} \to Q \to 0$$

we have the following Eagon-Northcott complex,

$$0 \to S^2S^* \to S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \to \bigwedge^2 H^0(\vartheta_{\mathbb{P}^n}) \otimes \vartheta_{\mathbb{G}} \to \vartheta_{\mathbb{G}}(1) \to 0$$

Let changes a little that sequence to have the one we need

$$0 \to S^2S^* \otimes \vartheta_{\mathbb{G}}(-1) \to S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) \to \bigwedge^2 H^0(\vartheta_{\mathbb{P}^n}) \otimes \vartheta_{\mathbb{G}}(-1) \to \vartheta_{\mathbb{G}} \to 0$$

The last thing we have to do with this sequence is to complete it with the Euler exact sequence:

$$0 \to \Omega_{\mathbb{P}|\mathbb{G}} \to \vartheta_{\mathbb{P}|\mathbb{G}}(-1)^{\binom{n+1}{2}} = \vartheta_{\mathbb{G}}(-1) \otimes \bigwedge^2 H^0(\vartheta_{\mathbb{P}^n}(1)) \to \vartheta_{\mathbb{G}} \to 0$$

so we have

$$0 \quad \to \quad S^2S^* \otimes \vartheta_{\mathbb{G}}(-1) \quad \to \quad S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) \quad \ldots$$

$$\begin{array}{ccccccccc}
\dots \quad S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) & & \longrightarrow & & \bigwedge^2 H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) & \to & \vartheta_{\mathbb{G}} & \to & 0 \\
 & \searrow & & \nearrow & & & & & \\
 & & \Omega_{\mathbb{P}|\mathbb{G}} & & & & & & \\
 & \nearrow & & \searrow & & & & & \\
0 & & & & 0 & & & &
\end{array}$$

Hence we have the following sequence

$$0 \to S^2 S^* \to S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \to \Omega_{\mathbb{P}|\mathbb{G}}(1) \to 0$$

and so

$$0 \to S^2 S^* \otimes \vartheta_{\mathbb{G}}(-1) \to S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) \to \Omega_{\mathbb{P}|\mathbb{G}} \to 0$$

We are going to use these sequences to build the following diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N^*_{G|P} & \longrightarrow & \Omega_{\mathbb{P}|\mathbb{G}} & \longrightarrow & \Omega_{\mathbb{G}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^* \otimes S^* \otimes \vartheta_G(-1) & \longrightarrow & S^* \otimes H^0(\vartheta_{P^n}(1)) \otimes \vartheta_G(-1) & \longrightarrow & S^* \otimes Q^* & \longrightarrow & 0 \\
 & & & & \uparrow & & & & \\
 & & & & (S^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1) & & & & \\
 & & & & \uparrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

If we prove that

$$\begin{array}{ccc}
\Omega_{\mathbb{P}|\mathbb{G}} & \xrightarrow{\varphi_1} & \Omega_{\mathbb{G}} \\
\varphi_4 \uparrow & & \uparrow \varphi_2 \\
S^* \otimes H^0(\vartheta_{\mathbb{P}^n}(1)) \otimes \vartheta_{\mathbb{G}}(-1) & \xrightarrow{\varphi_3} & S^* \otimes Q^*
\end{array}$$

is conmutative, then we can affirm that the following diagram conmute and all the sequences are exact

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & N^*_{G|P} & \longrightarrow & \Omega_{\mathbb{P}|\mathbb{G}} & \longrightarrow & \Omega_{\mathbb{G}} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & S^* \otimes S^* \otimes \vartheta_G(-1) & \longrightarrow & S^* \otimes H^0(\vartheta_{P^n}(1)) \otimes \vartheta_G(-1) & \longrightarrow & S^* \otimes Q^* & \longrightarrow & 0 \\
 & & \gamma \uparrow & & \uparrow & & & & \\
 & & (S^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1) & \overset{=}{\longleftrightarrow} & (S^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1) & & & & \\
 & & \uparrow & & \uparrow & & & & \\
 & & 0 & & 0 & & & &
\end{array}
$$

by the Snake Lemma.

Moreover, we easily see that γ is natural that map $(S^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1)$ in itself, that way we obteined the wedge $(\bigwedge^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1)$. So finally we get an expression for the normal and conormal bundle:

Teorema 3.2. *Let $\mathbb{G}(1, n)$ the grassmannian space of lines and let $N_{\mathbb{G}|\mathbb{P}}$ the normal bundle of $\mathbb{G}$ on $\mathbb{P}$. We have that:*

$$N^*_{\mathbb{G}|\mathbb{P}} = (\bigwedge^2 S^*) \otimes \vartheta_{\mathbb{G}}(-1)$$

$$N_{\mathbb{G}|\mathbb{P}} = (\bigwedge^2 S^*) \otimes \vartheta_{\mathbb{G}}(1)$$